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Random Search for Zeroes

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1. Introduction

Let f be a continuous real-valued function defined on the interval [0, 1], with the properties that $f(0) > 0 > f(1)$, and that f possesses a unique zero α, α unknown, $0 < \alpha < 1$. A standard way of approximating α is the bisection method. One tests the sign of $f(\frac{1}{2})$; if this is positive, then $\frac{1}{2} < \alpha < 1$ and one next tests $f(\frac{3}{4})$, if negative, then one tests $f(\frac{1}{4})$ etc. In this way, the zero α can be localized to a known interval of length 2^{-n} after n steps, by successively selecting the midpoints of the appropriate intervals.

Now suppose that instead of selecting the midpoint at each stage, we pick a point 'at random,' in accordance with a given distribution. Then we can ask for the expected value of the length of interval known to contain α after n steps. In this paper, we find an iterative formula for this and similar quantities, and investigate the behaviour for large n. For the solution of a related question in the same context, we refer to [1].

Let F be a monotonic function defined on the interval [0, 1] such that $F(0) = 0$ and $F(1) = 1$. For reasons which will appear later, we assume that F is absolutely continuous. F is to be used as the distribution function on [0, 1].

The basic assumption is that $x_1, x_2, x_3, \ldots, x_n, \ldots$ is a sequence of independently distributed 'random variables' satisfying the inequalities $l_n \leqslant x_{n+1} \leqslant r_n$, $n = 0, 1, 2, \ldots$ where $l_0 = 0$, $r_0 = 1$, and

$$[l_{n+1}, r_{n+1}] = \begin{cases} [l_n, x_{n+1}] & \text{if} \quad x_{n+1} > \alpha \\ [x_{n+1}, r_n] & \text{if} \quad x_{n+1} < \alpha \end{cases}$$

$$l_{n+1} = r_{n+1} = x_{n+1} \quad \text{if} \quad x_{n+1} = \alpha.$$

We assume further that if $l_n \leqslant a < b \leqslant r_n$, then

$$\text{prob}\,\{a \leqslant x_{n+1} \leqslant b\} = F\left(\frac{b - l_n}{r_n - l_n}\right) - F\left(\frac{a - l_n}{r_n - l_n}\right),$$

and if $l_n \leqslant a = b \leqslant r_n$, then

$$\text{prob}\,\{a = x_{n+1} = b\} = 1.$$

It should be noted that the distribution function is redefined for each subinterval of [0, 1]. It is evident that for all n,

$$0 \leqslant l_n \leqslant l_{n+1} \leqslant \alpha \leqslant r_{n+1} \leqslant r_n \leqslant 1.$$

It is convenient to define the function $F(\xi, s, t, \eta)$ for

$$0 \leqslant \xi \leqslant s \leqslant \alpha \leqslant t \leqslant \eta \leqslant 1,$$

by the equations

$$\begin{aligned} F(\xi, s, t, \eta) &= 1 - F\left(\frac{t-\xi}{\eta-\xi}\right) + F\left(\frac{s-\xi}{\eta-\xi}\right) \qquad &\text{if} \quad & \xi < \eta \\ &= 1 &\text{if} \quad & \xi = s = t = \eta. \end{aligned}$$

Two properties of $F(\xi, s, t, \eta)$ are needed later. They are

$$F(\xi, s, t, \eta) = F(\xi + c, s + c, t + c, \eta + c) \tag{1}$$

and

$$F(\xi, s, t, \eta) = F(c\xi, cs, ct, c\eta) \tag{2}$$

for suitable constants c.

2. Joint Probability

Of basic importance in all subsequent calculations is the joint probability distribution of the end points of the interval $[l_n, r_n]$ which contains the zero α.

Definition. *$G_n(s, t)$ is the probability that both $l_n < s$ and $t < r_n$; where $0 \leqslant s \leqslant \alpha \leqslant t \leqslant 1$, and n is a non-negative integer.*

Remark. $G_n(0, t) = G_n(s, 1) = 0$, and $G_n(\alpha, \alpha) = 1$ for all n. $G_0(s, t) = 1$ if both $s = 0$ and $t = 1$.

The fundamental recurrence relation for $G_n(s, t)$ is given by the following theorem.

Theorem 1. *If $0 < s \leqslant t < 1$, then for each non-negative integer k,*

$$G_{k+1}(s, t) = \iint_R F(\xi, s, t, \eta)\, dG_k(\xi, \eta) \tag{3}$$

where R is the rectangle $0 \leqslant \xi \leqslant s$, $t \leqslant \eta \leqslant 1$. The sense in which the integral is to be taken is the standard Riemann-Stieltjes integral as given below.

PROOF. For any $\epsilon > 0$, ϵ sufficiently small, take partitions

$$0 = y_0 < y_1 < y_2 < \cdots < y_n = s,$$

$$t = z_0 < z_1 < z_2 < \cdots < z_m = 1$$

such that

$$\sup_{i=1}^{n} | y_i - y_{i-1} | < \epsilon \qquad \text{and} \qquad \sup_{j=1}^{m} | z_j - z_{j-1} | < \epsilon.$$

By mutual exclusion, the probability that both

$$y_{i-1} \leqslant l_k < y_i \qquad \text{and} \qquad z_{j-1} < r_k \leqslant z_j$$

is

$$G_k(y_i, z_{j-1}) + G_k(y_{i-1}, z_j) - G_k(y_i, z_j) - G_k(y_{i-1}, z_{j-1}).$$

Moreover, given that $l_k < s$ and $t < r_k$, the probability that both $l_{k+1} < s$ and $t < r_{k+1}$ is the probability that either $l_k \leqslant x_{k+1} < s$ or $t < x_{k+1} \leqslant r_k$, and this equals $F(l_k, s, t, r_k)$. It follows, by adding over all pairs of subintervals in the partitions and letting $\epsilon \to 0$, that

$$G_{k+1}(s, t) = \lim \sum_{i=1}^{n} \sum_{j=1}^{m} F(\xi_i, s, t, \eta_j)$$
$$\times [G_k(y_i, z_{j-1}) + G_k(y_{i-1}, z_j) - G_k(y_i, z_j) - G_k(y_{i-1}, z_{j-1})], \tag{4}$$

where ξ_i and η_j are arbitrary points satisfying

$$y_{i-1} \leqslant \xi_i < y_i \qquad \text{and} \qquad z_{j-1} < \eta_j \leqslant z_j .$$

This limit, which defines the integral in the statement of the theorem, exists since $F(\xi, s, t, \eta)$ is a continuous function of ξ and η, for fixed s and t, and the variation of G_k over R is at most one, as can be shown easily.

LEMMA 1. *If* $0 < s < t < 1$, *and* k *is a non-negative integer, then*

$$G_{k+1}(s, t) = - \int_{\xi=0}^{s} G_k(\xi, t)\, dF(\xi, s, t, t)$$
$$+ \int_{\eta=t}^{1} G_k(s, \eta)\, dF(s, s, t, \eta) + \iint_R G_k(\xi, \eta)\, dF(\xi, s, t, \eta). \tag{5}$$

PROOF. Using the facts that $y_0 = 0$, $y_n = s$, $z_0 = t$, $z_m = 1$, and

$G_k(0, t) = G_k(s, 1) = 0$, the approximating Riemann sum (4) can be rearranged to give

$$F(\xi_n, \eta_1) G_k(s, t) + \sum_{i=1}^{n-1} G_k(y_i, t) [F(\xi_i, \eta_1) - F(\xi_{i+1}, \eta_1)]$$

$$+ \sum_{j=1}^{m-1} G_k(s, z_j) [F(\xi_n, \eta_{j+1}) - F(\xi_n, \eta_j)]$$

$$+ \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} G_k(y_i, z_j) [F(\xi_i, \eta_{j+1}) + F(\xi_{i+1}, \eta_j)$$

$$- F(\xi_i, \eta_j) - F(\xi_{i+1}, \eta_{j+1})],$$

where, for brevity, we put $F(\xi, \eta) = F(\xi, s, t, \eta)$. Since F is continuous and G_k is of bounded variation, the result follows on letting $\epsilon \to 0$.

We can now derive a third recurrence relation for $G_k(s, t)$. We claim that for all integers $k \geqslant 0$,

$$G_{k+1}(s, t) = \int_0^s G_k\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right) dF(x) + \int_t^1 G_k\left(\frac{s}{x}, \frac{t}{x}\right) dF(x), \tag{6}$$

where $0 \leqslant s \leqslant t \leqslant 1$, and the integrals are to be taken in the ordinary Riemann-Stieltjes sense. A direct proof appears to be difficult even if possible, so we use a form of induction.

Let $A_k(s, t)$, $B_k(s, t)$, and $C_k(s, t)$ denote respectively the first, second, and third terms in the right side of (5), so that the result of Lemma 1 can be written

$$G_{k+1}(s, t) = A_k(s, t) + B_k(s, t) + C_k(s, t).$$

THEOREM 2. *Define the functions $H_k(s, t)$, for $k = 0, 1, 2,...$ and $0 \leqslant s \leqslant \alpha$, $\alpha \leqslant t \leqslant 1$, by the equations*

$$H_0(0, t) = H_0(s, 1) = 0,$$

$$H_0(s, t) = 1 \qquad \text{if} \qquad 0 < s \leqslant t < 1,$$

$$H_{k+1}(s, t) = \int_0^s H_k\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right) dF(x) + \int_t^1 H_k\left(\frac{s}{x}, \frac{t}{x}\right) dF(x).$$

Then $H_k(s, t) = G_k(s, t)$ for each integer $k \geqslant 0$.

PROOF. The theorem is trivial if $k = 0$. If $k = 1$, then, using the definitions and (4),

$$H_1(s, t) = 1 - F(t) + F(s),$$

and

$$G_1(s, t) = \lim F(\xi_1, s, t, \eta_m)\, G_0(y_1, z_{m-1})$$
$$= F(0, s, t, 1) = 1 - F(t) + F(s).$$

Assume the induction hypothesis that for some integer $k \geqslant 0$,

$$G_k(s, t) = H_k(s, t) \qquad \text{and} \qquad G_{k+1}(s, t) = H_{k+1}(s, t).$$

The remainder of the proof falls into two parts. The first consists of showing that

$$C_{k+1}(s, t) = \int_0^s C_k\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right) dF(x) + \int_t^1 C_k\left(\frac{s}{x}, \frac{t}{x}\right) dF(x).$$

The proof of this is omitted. It is a routine calculation using the induction hypothesis, a change of variable and a change of order of integration in the resulting repeated integrals, and Eqs. (1) and (2).

The second part of the proof commences with

$$\begin{aligned}
G_{k+2}(s, t) - C_{k+1}(s, t) &= A_{k+1}(s, t) + B_{k+1}(s, t) \\
&= -\int_{\xi=0}^{s} H_{k+1}(\xi, t)\, dF(\xi, s, t, t) \\
&\quad + \int_{\eta=t}^{1} H_{k+1}(s, \eta)\, dF(s, s, t, \eta) \\
&= -\int_{\xi=0}^{s}\int_{x=0}^{\xi} G_k\left(\frac{\xi-x}{1-x}, \frac{t-x}{1-x}\right) dF(x)\, dF(\xi, s, t, t) \\
&\quad - \int_{\xi=0}^{s}\int_{x=t}^{1} G_k\left(\frac{\xi}{x}, \frac{t}{x}\right) dF(x)\, dF(\xi, s, t, t) \\
&\quad + \int_{\eta=t}^{1}\int_{x=0}^{s} G_k\left(\frac{s-x}{1-x}, \frac{\eta-x}{1-x}\right) dF(x)\, dF(s, s, t, \eta) \\
&\quad + \int_{\eta=t}^{1}\int_{x=\eta}^{1} G_k\left(\frac{s}{x}, \frac{\eta}{x}\right) dF(x)\, dF(s, s, t, \eta),
\end{aligned}$$

which follows from Lemma 1 and the induction hypothesis. By a change of variable and a change of order of integration in each integral, and using the result of the first part of the proof, the right-hand side simplifies to

$$\int_0^s H_{k+1}\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right) dF(x) + \int_t^1 H_{k+1}\left(\frac{s}{x}, \frac{t}{x}\right) dF(x) - C_{k+1}(s, t),$$

which yields

$$G_{k+2}(s, t) = H_{k+2}(s, t).$$

The functions $H_k(s, t)$ and $G_k(s, t)$ are all nonnegative and hence absolutely integrable over R, by induction on k. This justifies the changes of order in integration. The changes of variable used are given by mappings of the type

$$u = \frac{s - x}{1 - x}, \qquad v = \frac{t - x}{1 - x} \qquad \text{and} \qquad u = \frac{s}{x}, \qquad v = \frac{t}{x}.$$

Finally, the particular cases $0 < s = t < 1$, $0 = s = t$, $s = t = 1$, $0 = s < t \leqslant 1$, and $0 \leqslant s < t = 1$, can all be verified directly, so that (6) is true for all s and t satisfying $0 \leqslant s \leqslant t \leqslant 1$. This completes the proof.

3. Moments

We are now in a position to compute moments and expected values of various functions of l_n and r_n. In particular, we can develop formulas for the expected value and variance of $r_n - l_n$. Clearly, these will be functions of the zero α. For the remainder of this paper, E will denote expected value, and R will denote the rectangle $\{(x, y) : 0 \leqslant x \leqslant \alpha, \alpha \leqslant y \leqslant 1\}$.

We employ the following notation:

$$y_n(\alpha) = E(l_n) = \iint_R x \, dG_n(x, y)$$

$$z_n(\alpha) = E(r_n) = \iint_R y \, dG_n(x, y)$$

$$h_n(\alpha) = E(l_n^2) = \iint_R x^2 \, dG_n(x, y)$$

$$k_n(\alpha) = E(r_n^2) = \iint_R y^2 \, dG_n(x, y)$$

$$p_n(\alpha) = E(l_n r_n) = \iint_R xy \, dG_n(x, y)$$

$$\begin{aligned}
e_n(\alpha) &= E(r_n - l_n) = z_n(\alpha) - y_n(\alpha) \\
w_n(\alpha) &= E(r_n + l_n) = z_n(\alpha) + y_n(\alpha) \\
f_n(\alpha) &= E(r_n - l_n)^2 = h_n(\alpha) + k_n(\alpha) - 2p_n(\alpha) \\
v_n(\alpha) &= f_n(\alpha) - [e_n(\alpha)]^2.
\end{aligned}$$

All the above Stieltjes Integrals are to be taken in the same sense as in Theorem 1.

We derive recurrence relations for some of the above functions.

THEOREM 3. *For each integer* $n \geqslant 0$,

$$y_{n+1}(\alpha) = \int_0^\alpha t\, dF(t) + \int_0^\alpha (1-t)\, y_n\left(\frac{\alpha - t}{1-t}\right) dF(t) + \int_\alpha^1 t y_n\left(\frac{\alpha}{t}\right) dF(t), \tag{7}$$

where, clearly, $y_0(\alpha) = 0$.

PROOF

$$\begin{aligned} y_n(\alpha) &= \iint_R x\, dG_n(x, y) \\ &= \int_0^\alpha x\, dG_n(x, \alpha) - \int_0^\alpha x\, dG_n(x, 1) \\ &= \int_0^\alpha x\, dG_n(x, \alpha) \qquad \text{since} \qquad G_n(x, 1) = 0, \\ &= \alpha - \int_0^\alpha G_n(x, \alpha)\, dx \qquad \text{by parts.} \end{aligned}$$

So

$$\begin{aligned} \alpha - y_{n+1}(\alpha) &= \int_0^\alpha G_{n+1}(x, \alpha)\, dx \\ &= \int_0^\alpha \int_0^x G_n\left(\frac{x-t}{1-t}, \frac{\alpha - t}{1-t}\right) dF(t)\, dx + \int_0^\alpha \int_\alpha^1 G_n\left(\frac{x}{t}, \frac{\alpha}{t}\right) dF(t)\, dx \\ &= \int_0^\alpha \int_0^{(\alpha-t)/(1-t)} (1-t)\, G_n\left(u, \frac{\alpha-t}{1-t}\right) du\, dF(t) \\ &\qquad + \int_\alpha^1 \int_0^{\alpha/t} t G_n\left(u, \frac{\alpha}{t}\right) du\, dF(t) \\ &= \int_0^\alpha (1-t)\left[\frac{\alpha-t}{1-t} - y_n\left(\frac{\alpha-t}{1-t}\right)\right] dF(t) \\ &\qquad + \int_\alpha^1 t\left[\frac{\alpha}{t} - y_n\left(\frac{\alpha}{t}\right)\right] dF(t), \end{aligned}$$

from which the result follows.

Similarly, one can prove that

$$z_{n+1}(\alpha) = \int_0^\alpha t\, dF(t) + \int_0^\alpha (1-t)\, z_n\left(\frac{\alpha-t}{1-t}\right) dF(t) + \int_\alpha^1 t z_n\left(\frac{\alpha}{t}\right) dF(t). \tag{8}$$

It follows that for all integers $n \geqslant 0$, $e_0(\alpha) = 1$, and

$$e_{n+1}(\alpha) = \int_0^\alpha (1-t)\, e_n\left(\frac{\alpha-t}{1-t}\right) dF(t) + \int_\alpha^1 t e_n\left(\frac{\alpha}{t}\right) dF(t). \tag{9}$$

Similarly, $w_0(\alpha) = 1$, and

$$w_{n+1}(\alpha) = 2\int_0^\alpha t\,dF(t) + \int_0^\alpha (1-t)\,w_n\left(\frac{\alpha - t}{1-t}\right) dF(t) + \int_\alpha^1 tw_n\left(\frac{\alpha}{t}\right) dF(t) \tag{10}$$

By a similar, but more complicated calculation, it can be shown that $h_0(\alpha) = 0$, $k_0(\alpha) = 1$,

$$h_{n+1}(\alpha) = \int_0^\alpha t^2\,dF(t) + 2\int_0^\alpha t(1-t)\,y_n\left(\frac{\alpha - t}{1-t}\right) dF(t)$$
$$+ \int_0^\alpha (1-t)^2\,h_n\left(\frac{\alpha - t}{1-t}\right) dF(t) + \int_\alpha^1 t^2h_n\left(\frac{\alpha}{t}\right) dF(t), \tag{11}$$

and

$$k_{n+1}(\alpha) = \int_0^\alpha t^2\,dF(t) + 2\int_0^\alpha t(1-t)\,z_n\left(\frac{\alpha - t}{1-t}\right) dF(t)$$
$$+ \int_0^\alpha (1-t)^2\,k_n\left(\frac{\alpha - t}{1-t}\right) dF(t) + \int_\alpha^1 t^2k_n\left(\frac{\alpha}{t}\right) dF(t). \tag{12}$$

To evaluate the product moment $p_n(\alpha)$, a different method is indicated. First, it is easy to show, using (6), that if

$$q_n(\alpha) = \iint_R G_n(x, y)\,dx\,dy,$$

then

$$q_{n+1}(\alpha) = \int_0^\alpha (1-t)^2\,q_n\left(\frac{\alpha - t}{1-t}\right) dF(t) + \int_\alpha^1 t^2q_n\left(\frac{\alpha}{t}\right) dF(t).$$

Next, by considering the approximating Riemann sum for $p_n(\alpha)$, and rearranging terms as in Lemma 1, one can show that

$$p_n(\alpha) = -\int_\alpha^1\int_0^\alpha G_n(x, y)\,dx\,dy + \int_\alpha^1 \alpha G_n(\alpha, y)\,dy - \int_0^\alpha \alpha G_n(x, \alpha)\,dx + \alpha^2.$$

It follows that

$$\begin{aligned} p_n(\alpha) &= -q_n(\alpha) + \alpha[z_n(\alpha) - \alpha] - \alpha[\alpha - y_n(\alpha)] + \alpha^2 \\ &= \alpha w_n(\alpha) - q_n(\alpha) - \alpha^2. \end{aligned}$$

By using the recurrence relations for $w_n(\alpha)$ and $q_n(\alpha)$, it is now easy to show that $p_0(\alpha) = 0$, and

$$p_{n+1}(\alpha) = \int_0^\alpha t^2\, dF(t) + \int_0^\alpha t(1-t)\, w_k\left(\frac{\alpha - t}{1 - t}\right) dF(t) + \int_0^\alpha (1-t)^2\, p_n\left(\frac{\alpha - t}{1 - t}\right) dF(t) + \int_\alpha^1 t^2 p_n\left(\frac{\alpha}{t}\right) dF(t), \tag{13}$$

for all integers $n \geqslant 0$. From this result, together with (11) and (12), it follows that $f_0(\alpha) = 1$, and

$$f_{n+1}(\alpha) = \int_0^\alpha (1-t)^2 f_n\left(\frac{\alpha - t}{1 - t}\right) dF(t) + \int_\alpha^1 t^2 f_n\left(\frac{\alpha}{t}\right) dF(t). \tag{14}$$

4. Asymptotic Results

We attempt now to investigate the behaviour of $e_n(\alpha)$ and $v_n(\alpha)$ as $n \to \infty$, and to show how this depends on the distribution function F. The main result is that $e_n(\alpha) \sim \lambda^n \bar{e}(\alpha)$, where λ is a constant depending on F alone, and $\bar{e}(\alpha)$ is a solution to a homogeneous linear integral equation.

We define two numbers λ and μ by the equations

$$\lambda = \int_0^1 (1-t)^2\, dF(t) + \int_0^1 t^2\, dF(t) \tag{15}$$

and

$$\mu = \int_0^1 (1-t)^3\, dF(t) + \int_0^1 t^3\, dF(t). \tag{16}$$

A straightforward calculation using (9) and (14) shows that

$$\int_0^1 e_n(\alpha)\, d\alpha = \lambda^n \qquad \text{and} \qquad \int_0^1 f_n(\alpha)\, d\alpha = \mu^n, \tag{17}$$

for all integers $n \geqslant 0$. Thus 'globally,' e_n and f_n decrease like convergent geometric sequences. Bounds for λ and μ are given by the estimates

$$\lambda = \int_0^1 t^2 + (1-t)^2\, dF(t) < \int_0^1 dF(t) = 1,$$

$$\lambda = \int_0^1 t^2 + (1-t)^2\, dF(t) > \tfrac{1}{2} \int_0^1 dF(t) = \tfrac{1}{2},$$

and $\lambda^2 < \mu < \lambda$.

The fact that $\lambda > \frac{1}{2}$ shows that one cannot expect to have a faster rate of convergence to α than that provided by the bisection method. λ can assume any value between $\frac{1}{2}$ and 1, as shown by the symmetric B-distribution for which $F'(t) = \phi(t) = t^{m-1}(1-t)^{m-1}/B(m, m)$, $m > 0$, and $\lambda = (m+1)/(2m+1)$. It is noteworthy that the speed of 'convergence λ depends only on the first two moments of F.

It is convenient to normalize the functions $e_n(\alpha)$, by defining a new sequence $\bar{e}_n(\alpha) = \lambda^{-n} e_n(\alpha)$. We have $\bar{e}_0(\alpha) = 1$,

$$\int_0^1 \bar{e}_n(\alpha)\, d\alpha = 1 \tag{18}$$

and

$$\lambda \bar{e}_{n+1}(\alpha) = \int_0^\alpha (1-t)\, \bar{e}_n\left(\frac{\alpha - t}{1 - t}\right) dF(t) + \int_\alpha^1 t \bar{e}_n\left(\frac{\alpha}{t}\right) dF(t), \tag{19}$$

for all integers $n \geqslant 0$.

The problem of showing that $e_n(\alpha) \sim \lambda^n \bar{e}(\alpha)$ can now be solved by considering the existence of $\lim_{n\to\infty} \bar{e}_n(\alpha)$ for each α.

For the remainder of this paper, we assume that $\phi(t) = F'(t)$ exists almost everywhere, $\phi(t)$ is essentially bounded, with ess-sup $\phi(t) \leqslant M$, and $\phi(t) = \phi(1-t)$. This last assumption simplifies calculations, without destroying the validity of the subsequent results. Three consequences are

$$\int_0^1 t\, dF(t) = \tfrac{1}{2},$$

$$\lambda = 2\int_0^1 t^2\, dF(t), \qquad \text{and} \qquad \mu = 2\int_0^1 t^3\, dF(t).$$

Moreover, the functions $\bar{e}_n(\alpha)$ are all symmetric about $\alpha = \frac{1}{2}$.

Lemma 2. *The sequence $\bar{e}_n(\alpha)$ is uniformly bounded. More precisely, there exists a real number β, $0 < \beta < \frac{1}{2}$, such that*

$$\bar{e}_n(\alpha) \leqslant \frac{M}{\lambda}\left(\frac{1}{\beta} + \frac{1}{1-\beta}\right), \qquad 0 \leqslant \alpha \leqslant 1,$$

for all integers $n \geqslant 0$.

Proof. The function

$$g(\alpha) = \int_\alpha^1 u\, dF(u) + \int_0^\alpha (1-u)\, dF(u)$$

is uniformly continuous in α, and has the value $\frac{1}{2}$ if α is 0 or 1. Thus, since $\lambda > \frac{1}{2}$, there exists β, $0 < \beta < \frac{1}{2}$, such that if either $0 \leqslant \alpha \leqslant \beta$ or $1 - \beta \leqslant \alpha \leqslant 1$, then $g(\alpha) \leqslant \lambda$.

Consider two cases. If $\beta \leqslant \alpha \leqslant 1 - \beta$, then by a change of variable, and by using the fact that ϕ is symmetric,

$$\lambda \bar{e}_{n+1}(\alpha) = (1 - \alpha)^2 \int_{1-\alpha}^{1} u^{-3} \bar{e}_n(u)\, \phi\left(\frac{1-\alpha}{u}\right) du + \alpha^2 \int_{\alpha}^{1} u^{-3} \bar{e}_n(u)\, \phi\left(\frac{\alpha}{u}\right) du$$

$$\leqslant M \left[\frac{1}{1-\alpha} \int_{1-\alpha}^{1} \bar{e}_n(u)\, du + \frac{1}{\alpha} \int_{\alpha}^{1} \bar{e}_n(u)\, du\right]$$

$$\leqslant M \left(\frac{1}{1-\alpha} + \frac{1}{\alpha}\right)$$

by (18). So the result holds for all integers $n \geqslant 1$, for the case $\beta \leqslant \alpha \leqslant 1 - \beta$. Since $\bar{e}_0(\alpha) = 1$, the case $n = 0$ is also satisfied.

For the remaining values of α, we can use induction together with the above. If, for some integer k, the required inequality holds, and if $0 \leqslant \alpha < \beta$, then

$$\lambda \bar{e}_{k+1}(\alpha) = \int_0^{\alpha} (1 - t)\, \bar{e}_k\left(\frac{\alpha - t}{1 - t}\right) dF(t) + \int_{\alpha}^{1} t \bar{e}_k\left(\frac{\alpha}{t}\right) dF(t)$$

$$\leqslant \frac{M}{\lambda}\left(\frac{1}{\beta} + \frac{1}{1-\beta}\right) g(\alpha) \qquad \text{by the induction hypothesis,}$$

$$\leqslant M\left(\frac{1}{\beta} + \frac{1}{1-\beta}\right).$$

Since $\bar{e}_{k+1}(\alpha)$ is symmetric, the proof is complete.

We wish to show that $\bar{e}_n(\alpha)$ is a Cauchy sequence for each α. To this end, we define $\delta_n(\alpha) = \bar{e}_{n+p}(\alpha) - \bar{e}_n(\alpha)$, where p is an arbitrary positive integer, and put

$$B_{n,k} = \int_0^1 t^k\, \delta_n(t)\, dt.$$

Lemma 3. $\lim_{n\to\infty} B_{n,k} = 0$, *for all integers* $k \geqslant 0$.

Proof. Since $\mu/\lambda < 1$, we can choose γ such that $\mu/\lambda < \gamma < 1$. Define the sequence A_k by the equations $A_0 = 1$, and

$$A_k = 4 + \left[\frac{\lambda}{(\lambda\gamma - \mu)}\right] \sum_{r=0}^{k-1} \binom{k}{r} A_r,$$

where $\binom{k}{r}$ is the usual binomial coefficient.

We prove by induction on k that $| B_{n,k} | \leqslant A_k \gamma^n$. If $k = 0$,

$$B_{n,0} = \int_0^1 \delta_n(t)\, dt$$

$$= \int_0^1 \bar{e}_{n+p}(t)\, dt - \int_0^1 \bar{e}_n(t)\, dt$$

$$= 1 - 1 = 0.$$

Assume now that the result is proved for all integers r, $0 \leqslant r \leqslant k - 1$, where $k \geqslant 1$. By linearity and (19),

$$\lambda \delta_{n+1}(\alpha) = \int_0^\alpha (1 - t)\, \delta_n\left(\frac{\alpha - t}{1 - t}\right) dF(t) + \int_\alpha^1 t\delta_n\left(\frac{\alpha}{t}\right) dF(t).$$

Multiplying each side by α^k and integrating, gives

$$\lambda B_{n+1,k} = \int_0^1 \int_0^\alpha \alpha^k (1 - t)\, \delta_n\left(\frac{\alpha - t}{1 - t}\right) dF(t)\, d\alpha + \int_0^1 \int_\alpha^1 \alpha^k t\delta_n\left(\frac{\alpha}{t}\right) dF(t)\, d\alpha,$$

which implies, on putting $1 - u = t$ and $1 - v = (\alpha - t)/(1 - t)$ in the first integral, and $u = t$ and $v = \alpha/t$ in the second, that

$$\lambda B_{n+1,k} = \int_0^1 \int_0^1 (1 - uv)^k\, u^2 \delta_n(v)\, dF(u)\, dv + \int_0^1 \int_0^1 u^{k+2} v^k \delta_n(v)\, dF(u)\, dv$$

$$= \sum_{r=0}^{k} \binom{k}{r} (-1)^r B_{n,r} \int_0^1 u^{r+2}\, dF(u) + B_{n,k} \int_0^1 u^{k+2}\, dF(u)$$

$$= [1 + (-1)^k]\, B_{n,k} \int_0^1 u^{k+2}\, dF(u)$$

$$+ \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r B_{n,r} \int_0^1 u^{r+2}\, dF(u).$$

Hence,

$$\lambda\, | B_{n+1,k} | \leqslant 2\, | B_{n,k} | \int_0^1 u^3\, dF(u) + \sum_{r=0}^{k-1} \binom{k}{r} | B_{n,r} | \int_0^1 u^2\, dF(u)$$

$$= \mu\, | B_{n,k} | + \tfrac{1}{2} \lambda \sum_{r=0}^{k-1} \binom{k}{r} | B_{n,r} | .$$

By the induction hypothesis, it follows that

$$
\begin{aligned}
| B_{n+1,k} | &\leqslant \frac{\mu}{\lambda} | B_{n\,k} | + \frac{1}{2} \gamma^n \sum_{r=0}^{k-1} \binom{k}{r} A_k \\
&= \frac{\mu}{\lambda} | B_{n,k} | + \frac{1}{2} \gamma^n (A_k - 4) \left(\gamma - \frac{\mu}{\lambda}\right),
\end{aligned}
$$

and hence, by considering this inequality for all values of n from 0 upwards, that

$$
\begin{aligned}
| B_{n+1\,k} | &\leqslant \left(\frac{\mu}{\lambda}\right)^{n+1} | B_{0,k} | + \frac{1}{2} \left(\gamma - \frac{\mu}{\lambda}\right) (A_k - 4) \sum_{m=0}^{n} \gamma^m \left(\frac{\mu}{\lambda}\right)^{n-m} \\
&\leqslant \gamma^{n+1} | B_{0,k} | + \tfrac{1}{2} (A_k - 4) \gamma^{n+1} \\
&\leqslant \gamma^{n+1} [2 + \tfrac{1}{2} (A_k - 4)] \leqslant \gamma^{n+1} A_k ,
\end{aligned}
$$

where the inequality $| B_{0,k} | \leqslant 2$ arises from the inequality

$$
| B_{0,k} | = \left| \int_0^1 t^k \delta_0(t)\, dt \right| \leqslant \int_0^1 | \delta_0(t) |\, dt \leqslant 2.
$$

This completes the induction.

COROLLARY 1. *For any polynomial $p(t)$,*

$$
\lim_{n\to\infty} \int_0^1 p(t)\, \delta_n(t)\, dt = 0.
$$

This is trivial by linearity.

COROLLARY 2. *For any continuous function $\psi(t)$ defined on* $[0, 1]$,

$$
\lim_{n\to\infty} \int_0^1 \psi(t)\, \delta_n(t)\, dt = 0.
$$

This follows by the Weierstrass approximation theorem and Corollary 1.

COROLLARY 3. *For any function $\chi(t) \in L_1[0, 1]$,*

$$
\lim_{n\to\infty} \int_0^1 \chi(t)\, \delta_n(t)\, dt = 0.
$$

PROOF. For any $\epsilon > 0$, there exists a continuous function $\psi(t)$ such that

$$
\int_0^1 | \chi(t) - \psi(t) |\, dt < \epsilon.
$$

But then,

$$\left| \int_0^1 \chi(t)\, \delta_n(t)\, dt \right| \leqslant \int_0^1 | \chi(t) - \psi(t) | \, | \delta_n(t) | \, dt + \left| \int_0^1 \psi(t)\, \delta_n(t)\, dt \right|$$

$$\leqslant E \int_0^1 | \chi(t) - \psi(t) | \, dt + \left| \int_0^1 \psi(t)\, \delta_n(t)\, dt \right|$$

$$\leqslant E\epsilon + \left| \int_0^1 \psi(t)\, \delta_n(t)\, dt \right| ,$$

where E is an upper bound for $\delta_n(t)$, which exists by Lemma 2. The result follows on letting $n \to \infty$, by Corollary 2.

THEOREM 4. *$\delta_n(\alpha)$ converges to 0 for all α.*

PROOF. The cases $\alpha = 0$ and $\alpha = 1$, are covered by the equation

$$\lambda \delta_{n+1}(0) = \delta_n(0) \int_0^1 t\phi(t)\, dt = \tfrac{1}{2}\, \delta_n(0),$$

which follows from (9) and the definition of δ_n. If $0 < \alpha < 1$, then

$$\lambda \delta_{n+1}(\alpha) = \alpha^2 \int_0^1 t^{-3} \chi_{[\alpha,1]}(t)\, \phi\left(\frac{\alpha}{t}\right) \delta_n(t)\, dt$$
$$+ (1-\alpha)^2 \int_0^1 t^{-3} \chi_{[1-\alpha,1]}(t)\, \phi\left(\frac{1-\alpha}{t}\right) \delta_n(t)\, dt,$$

where $\chi_{[a,b]}$ is the characteristic function of $[a, b]$. The coefficients of $\delta_n(t)$ in the integrands are L_1-integrable since ϕ is essentially bounded. The result follows by Corollary 3.

THEOREM 5. *$\lim_{n\to\infty} \bar{e}_n(\alpha) = \bar{e}(\alpha)$ exists for all α.*

PROOF. $\bar{e}_n(\alpha)$ is a uniformly bounded cauchy sequence for every α, since $\delta_n(\alpha) = \bar{e}_{n+p}(\alpha) - \bar{e}_n(\alpha)$, and p is arbitrary.

THEOREM 6. *$\bar{e}(\alpha)$, defined as in Theorem 5, satisfies the integral equation*

$$\lambda \bar{e}(\alpha) = \int_0^\alpha (1-t)\, \bar{e}\left(\frac{\alpha - t}{1-t}\right) \phi(t)\, dt + \int_\alpha^1 t \bar{e}\left(\frac{\alpha}{t}\right) \phi(t)\, dt. \qquad (20)$$

PROOF. Interchange of limit and integral in (19) is permissible by Lebesgue's dominated convergence theorem.

LEMMA 4. *If ϕ is continuous on $[0, 1]$, then $\bar{e}(\alpha)$ is continuous on $(0, 1)$.*

PROOF. A change of variable in (20) gives

$$\lambda\bar{e}(\alpha) = \alpha^2 \int_{\alpha}^{1} t^{-3}\bar{e}(t)\,\phi\left(\frac{\alpha}{t}\right) dt + (1-\alpha)^2 \int_{1-\alpha}^{1} t^{-3}\bar{e}(t)\,\phi\left(\frac{1-\alpha}{t}\right) dt$$

and the right-hand side of this equation is continuous.

If ϕ is N-times differentiable, then by using the standard results on differentiation under the integral sign, one can show that $\bar{e}(\alpha)$ is also N-times differentiable. It follows that $\bar{e}(\alpha)$ can be found by transforming (20) into a differential equation.

REMARK. Exactly the same methods (from Lemma 2 onwards) will give corresponding results for the sequence $f_n(\alpha)$.

REMARK. By a tedious calculation involving integral estimates, somewhat similar to the proof of Lemma 2, it can be shown that if ϕ is continuous, then the convergence of $\bar{e}_n(\alpha)$ to $\bar{e}(\alpha)$ is uniform on [0, 1]. Under the same assumption, every $\bar{e}_n(\alpha)$, and hence $\bar{e}(\alpha)$, is continuous on [0, 1].

5. RECTANGULAR DISTRIBUTION

As an example, we consider briefly the case $\phi(t) = 1$. It is easily checked that $\lambda = \frac{2}{3}$, and hence that

$$\tfrac{2}{3}\,\bar{e}(\alpha) = \alpha^2 \int_{\alpha}^{1} t^{-3}\bar{e}(t)\,dt + (1-\alpha)^2 \int_{1-\alpha}^{1} t^{-3}\bar{e}(t)\,dt$$

for $0 < \alpha < 1$. By differentiating twice, we obtain

$$2\alpha(1-\alpha)\,\bar{e}''(\alpha) + (1-2\alpha)\,\bar{e}'(\alpha) + 2\bar{e}(\alpha) = 0.$$

This has two linearly independent solutions only, given by

$$\bar{e}(\alpha) = k_1[\alpha(1-\alpha)]^{1/2} + k_2(1-2\alpha),$$

where k_1 and k_2 are constants. Since $\bar{e}(\alpha)$ is nonnegative and symmetric, k_2 must be zero. Since

$$\int_0^1 \bar{e}(\alpha)\,d\alpha = 1,$$

$k_1 = 8/\pi$, and so

$$\lim_{n\to\infty} \bar{e}_n(\alpha) = \frac{8}{\pi}\,[\alpha(1-\alpha)]^{1/2}$$

or

$$e_n(\alpha) \sim \frac{8}{\pi}\left(\frac{2}{3}\right)^n [\alpha(1-\alpha)]^{1/2}, \qquad 0 \leqslant \alpha \leqslant 1.$$

It is of interest to compute the variance $v_n(\alpha)$ for this case. It is easy to show by induction on n that

$$f_n(\alpha) = 3^{-n} + 6[2^{-n} - 3^{-n}]\,\alpha(1-\alpha).$$

Hence,

$$v_n(\alpha) = f_n(\alpha) - [e_n(\alpha)]^2 \sim 6\alpha\,\frac{1-\alpha}{2^n}.$$

The Complete Proof of Theorem 2

Proof. The theorem is trivial if $k = 0$. If $k = 1$, then, using the definitions and (4),

$$H_1(s, t) = 1 - F(t) + F(s),$$

and

$$\begin{aligned} G_1(s, t) &= \lim F(\xi_1, s, t, \eta_m)\, G_0(y_1, z_{m-1}) \\ &= F(0, s, t, 1) = 1 - F(t) + F(s). \end{aligned}$$

Assume the induction hypothesis that for some integer $k \geqslant 0$,

$$G_k(s, t) = H_k(s, t) \qquad \text{and} \qquad G_{k+1}(s, t) = H_{k+1}(s, t).$$

We prove under this assumption, first that

$$C_{k+1}(s, t) = \int_0^s C_k\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right) dF(x) + \int_t^1 C_k\left(\frac{s}{x}, \frac{t}{x}\right) dF(x)$$

and second that $G_{k+2}(s, t) = H_{k+2}(s, t)$. Now,

$$\begin{aligned} C_{k+1}(s, t) &= \iint_R G_{k+1}(\xi, \eta)\, dF(\xi, s, t, \eta) \\ &= \iint_R H_{k+1}(\xi, \eta)\, dF(\xi, s, t, \eta) \\ &= \int_{\eta=t}^1 \int_{\xi=0}^s \int_{x=0}^{\xi} H_k\left(\frac{\xi-x}{1-x}, \frac{\eta-x}{1-x}\right) dF(x)\, dF(\xi, s, t, \eta) \\ &\qquad + \int_{\eta=t}^1 \int_{\xi=0}^s \int_{x=\eta}^1 H_k\left(\frac{\xi}{x}, \frac{\eta}{x}\right) dF(x)\, dF(\xi, s, t, \eta) \end{aligned}$$

$$= \int_{x=0}^{s}\int_{\eta=t}^{1}\int_{\xi=x}^{s} H_k\left(\frac{\xi - x}{1 - x}, \frac{\eta - x}{1 - x}\right) dF(\xi, s, t, \eta)\, dF(x)$$

$$+ \int_{x=t}^{1}\int_{\xi=0}^{s}\int_{\eta=t}^{x} H_k\left(\frac{\xi}{x}, \frac{\eta}{x}\right) dF(\xi, s, t, \eta)\, dF(x)$$

$$= \int_0^s\int_{v=(t-x)/(1-x)}^{1}\int_{u=0}^{(s-x)/(1-x)} H_k(u, v)$$

$$\times\, dF(x + u - ux, s, t, x + v - vx)\, dF(x)$$

$$+ \int_t^1\int_{u=0}^{s/x}\int_{v=(t/x)}^{1} H_k(u, v)\, dF(ux, s, t, vx)\, dF(x)$$

$$= \int_0^s\int_{v=(t-x)/(1-x)}^{1}\int_{u=0}^{(s-x)/(1-x)} H_k(u, v)$$

$$\times\, dF\left(u, \frac{s - x}{1 - x}, \frac{t - x}{1 - x}, v\right) dF(x)$$

$$+ \int_t^1\int_{u=0}^{s/x}\int_{v=(t/x)}^{1} H_k(u, v)\, dF\left(u, \frac{s}{x}, \frac{t}{x}, v\right) dF(x)$$

by (1) and (2),

$$= \int_0^s C_k\left(\frac{s - x}{1 - x}, \frac{t - x}{1 - x}\right) dF(x) + \int_t^1 C_k\left(\frac{s}{x}, \frac{t}{x}\right) dF(x).$$

To complete the proof, we have, by Lemma 1,

$$G_{k+2}(s, t) - C_{k+1}(s, t) = A_{k+1}(s, t) + B_{k+1}(s, t)$$

$$= -\int_{\xi=0}^{s} H_{k+1}(\xi, t)\, dF(\xi, s, t, t)$$

$$+ \int_{\eta=t}^{1} H_{k+1}(s, \eta)\, dF(s, s, t, \eta)$$

$$= -\int_{\xi=0}^{s}\int_{x=0}^{\xi} G_k\left(\frac{\xi - x}{1 - x}, \frac{t - x}{1 - x}\right) dF(x)\, dF(\xi, s, t, t)$$

$$-\int_{\xi=0}^{s}\int_{x=t}^{1} G_k\left(\frac{\xi}{x}, \frac{t}{x}\right) dF(x)\, dF(\xi, s, t, t)$$

$$+\int_{\eta=t}^{1}\int_{x=0}^{s} G_k\left(\frac{s - x}{1 - x}, \frac{\eta - x}{1 - x}\right) dF(x)\, dF(s, s, t, \eta)$$

$$+\int_{\eta=t}^{1}\int_{x=\eta}^{1} G_k\left(\frac{s}{x}, \frac{\eta}{x}\right) dF(x)\, dF(s, s, t, \eta).$$

By a change of variable in each integral, and by a change in order of integration, the right-hand side becomes

$$
\begin{aligned}
&-\int_{v=(t-s)/(1-s)}^{t}\int_{u=0}^{(v-vs-t+s)/(1-t)} G_k(u, v) \\
&\qquad\qquad\qquad \times dF\left(\frac{t-v+u-ut}{1-v}, s, t, t\right) dF\left(\frac{t-v}{1-v}\right) \\
&-\int_{v=t}^{1}\int_{u=0}^{sv/t} G_k(u, v)\, dF\left(\frac{ut}{v}, s, t, t\right) dF\left(\frac{t}{v}\right) \\
&+\int_{u=0}^{s}\int_{v=(t-s+u-ut)/(1-s)}^{1} G_k(u, v)\, dF\left(s, s, t, \frac{s-u+v-vs}{1-u}\right) dF\left(\frac{s-u}{1-u}\right) \\
&+\int_{u=s}^{s/t}\int_{v=(ut)/s}^{1} G_k(u, v)\, dF\left(s, s, t, \frac{vs}{u}\right) dF\left(\frac{s}{u}\right) \\
&= \int_{v=(t-s)/(1-s)}^{t} A_k\left(\frac{v-vs-t+s}{1-t}, v\right) dF\left(\frac{t-v}{1-v}\right) \\
&\qquad + \int_{v=t}^{1} A_k\left(\frac{sv}{t}, v\right) dF\left(\frac{t}{v}\right) + \int_{u=0}^{s} B_k\left(u, \frac{t-s+u-ut}{1-s}\right) dF\left(\frac{s-u}{1-u}\right) \\
&\qquad + \int_{u=s}^{s/t} B_k\left(u, \frac{ut}{s}\right) dF\left(\frac{s}{u}\right) \\
&= \int_0^s A_k\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right) dF(x) + \int_t^1 A_k\left(\frac{s}{x}, \frac{t}{x}\right) dF(x) \\
&\qquad\qquad\qquad + \int_0^s B_k\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right) dF(x) \\
&\qquad\qquad\qquad + \int_t^1 B_k\left(\frac{s}{x}, \frac{t}{x}\right) dF(x) \\
&= \int_0^s \left[G_{k+1}\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right) - C_k\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right)\right] dF(x) \\
&\qquad\qquad\qquad + \int_t^1 \left[G_{k+1}\left(\frac{s}{x}, \frac{t}{x}\right) - C_k\left(\frac{s}{x}, \frac{t}{x}\right)\right] dF(x),
\end{aligned}
$$

which reduces by the first part of the theorem to

$$
\begin{aligned}
\int_0^s G_{k+1}\left(\frac{s-x}{1-x}, \frac{t-x}{1-x}\right) dF(x) + \int_t^1 G_{k+1}\left(\frac{s}{x}, \frac{t}{x}\right) dF(x) - C_{k+1}(s, t) \\
= H_{k+2}(s, t) - C_{k+1}(s, t).
\end{aligned}
$$

Hence, $G_{k+2}(s, t) = H_{k+2}(s, t)$ for $0 < s < t < 1$, as claimed.

The functions $H_k(s, t)$ and $G_k(s, t)$ are all non-negative and hence absolutely integrable over $0 \leqslant s \leqslant \alpha$, $\alpha \leqslant t \leqslant 1$, by induction on k. This justifies the changes of order in integration. Finally, the particular cases $0 < s = t < 1$, $0 = s = t$, $s = t = 1$, $0 = s < t \leqslant 1$, and $0 \leqslant s < t = 1$, can all be verified directly, so that (6) is true for all s and t satisfying $0 \leqslant s \leqslant t \leqslant 1$. This completes the proof.

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